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TORSION TENSOR FORMS ON INDUCED BUNDLES

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Abstract. Let ϕ be a map of a manifold M into another manifold N, $L(N)$ the bundle of all linear frames over N, and $\phi^{-1}(L(N))$ the bundle over M which is induced from ϕ and $L(N)$. Then, we construct a structure equation for the torsion form in $\phi^{-1}(L(N))$ which is induced from a torsion form in $L(N)$.

1. Introduction

Let $\phi : M \to N$ be a $C^\infty-$map between smooth manifolds M and N, $L(N)$ the bundle of all linear frames over N, and $\phi^{-1}(L(N)) =: Q$ the bundle which is induced from ϕ and $L(N)$. Then, the bundle homomorphism $\tilde{\phi} : Q \to L(N)$ between two principal fibre bundles $\phi^{-1}(TN) =: Q$ and $L(N)$ is defined by $\tilde{\phi}(u,x) = u$ $((u,x) \in Q, x \in M, \pi(u) = x, u \in L(N))$ ([1]).

Let Γ be an arbitrarily given connection in $L(N)$. Let ω and θ be the connection form and the canonical form in $L(N)$ which are defined from Γ ([1, 3]). Now, putting $\tilde{\phi}^*\omega =: \tilde{\omega}$ and $\tilde{\phi}^*\theta =: \tilde{\theta}$, we obtain the fact that $\tilde{\omega}$ and $\tilde{\theta}$ are a connection form and the canonical form for $\tilde{\omega}$ in $\phi^{-1}(TN)$ (cf. Lemma 2.1). And then, for $X \in \mathfrak{X}(M)$ we get the horizontal lift of X for the connection form $\tilde{\omega}$ in $\phi^{-1}(TN)$ (cf. Proposition 2.2).

Finally, we construct a structure equation for the torsion form $d\tilde{\theta} \circ h = \tilde{\Theta}$ in $\phi^{-1}(L(N))$ (cf. Theorem 2.6). These facts are very important in the study on the theory of connections. But, minute explications are rarely seen.

In this paper, we give full explanations for the above consequences.

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2. Torsion tensor forms on induced principal fibre bundles

Let M, N be two $C^\infty-$manifolds. We put $\dim M = m$ and $\dim N = n$. Let ϕ be a smooth map of M into N, $L(M)$ and $L(N)$ the bundles of linear frames over M and N, respectively. Let $\phi^{-1}(L(N))$ be the bundle induced by ϕ from $L(N)$. We concisely describe as follows :

$$\pi : L(N) \ni u \longmapsto \pi(u) \in N,$$
$$Q := \phi^{-1}(L(N)) := \{(u, x) \in L(N) \times M \mid x \in M,\ u \in \pi^{-1}(\{\phi(x)\})\},$$
$$(u, x)g := (ug, x) \quad ((u, x) \in Q, g \in GL(n; \mathbb{R})),$$
$$\tilde{u} : Q \ni (u, x) \longmapsto x \in M,$$

then $Q(M, GL(n; \mathbb{R}), \tilde{\pi})$ is the principal fibre bundle which is said to be the *induced bundle from ϕ and $L(N)$*.

Let $\tilde{\phi}$ be a map of $\phi^{-1}(L(N))$ into $L(N)$ which is defined by $\tilde{\phi}(u, x) = u$. Then $\tilde{\phi}$ is a homomorphism between two principal fibre bundles which is associated with the identity group isomorphism of the group $GL(n; \mathbb{R})$. In fact,

$$\tilde{\phi}((u, x)g) = \tilde{\phi}(ug, x) = ug = \tilde{\phi}(u, x) \cdot g \quad ((u, x) \in Q, g \in GL(n; \mathbb{R})).$$

Moreover, we have

$$\pi \circ \tilde{\phi} = \phi \circ \tilde{\pi}. \tag{2.1}$$

Let Γ be an arbitrarily given connection in $L(N)$, ω the connection form of Γ, and θ the canonical form (cf. [1]) of $L(N)$ defined by

$$\theta(X) = u^{-1}(\pi_*(X)) \qquad (X \in T_u(L(N)), u \in L(N)). \tag{2.2}$$

We put $\tilde{\omega} := \tilde{\phi}^*\omega$ and $\tilde{\theta} := \tilde{\phi}^*\theta$. Let $A^{\#}_{(u,x)}$ ($A \in \mathfrak{gl}(n; \mathbb{R})$, $(u, x) \in Q$) be the value of the fundamental vector field $A^{\#}$ at (u, x) which corresponds to A on Q. Similarly, let $A^{\#}_u$ be the value of the fundamental vector field $A^{\#}$ (cf. [2]) at $u \in L(N)$ which corresponds to A on $L(N)$.

For $X \in \mathfrak{X}(M)$, we put $\phi_*X =: \tilde{X}$. Let $\tilde{X}^*$ ($X \in \mathfrak{X}(M)$) be the horizontal lift of $\tilde{X}(= \phi_*X)$ for the connection form ω in $L(N)$. Then we have the following lemma.

Lemma 2.1.

(i) *$\tilde{\omega}$ is a connection form in the principal fibre bundle $Q(M, GL(n; \mathbb{R}), \tilde{\pi})$.*

(ii) *$\tilde{\theta}$ satisfies the following conditions :*

$$R_g^*\, \tilde{\theta} = Ad(g^{-1})\tilde{\theta} \qquad (g \in GL(n; \mathbb{R})),$$
$$\tilde{\theta}_{(u,x)}(A^{\#}) = 0 \qquad ((u, x) \in Q,\ A \in \mathfrak{gl}(n; \mathbb{R})),$$

where $Ad : GL(n;\mathbb{R}) \to GL(\mathfrak{gl}(n;\mathbb{R}))$ is the adjoint representation of $GL(n;\mathbb{R})$ in $\mathfrak{gl}(n;\mathbb{R})$.

Proof. For $(u,x) \in Q$, $(\tilde{X}^*, X) \in T_{(u,x)}Q$ $(X \in \mathfrak{X}(M))$, $g \in GL(n;\mathbb{R})$, and $A \in \mathfrak{gl}(n;\mathbb{R})$, by virtue of (2.1) we have the following properties ;

$$(R_g^* \, \tilde{\omega})(\tilde{X}^*, X) = \omega(R_{g^*}\tilde{X}^*) = Ad(g^{-1})(\tilde{\omega}(\tilde{X}^*, X)),$$

$$\tilde{\omega}(A^{\#}_{(u,x)}) = \tilde{\omega}\left(\lim_{t\to 0} \frac{(u\cdot \exp(tA), x) - (u,x)}{t}\right) = \omega(A^{\#}_u) = A,$$

$$\tilde{\theta}(A^{\#}_{(u,x)}) = \theta(A^{\#}_u) = 0,$$

$$(R_g^* \, \tilde{\theta})(\tilde{X}^*, X) = \tilde{\theta}(R_{g^*}\tilde{X}^*, X) = Ad(g^{-1}) \; \theta(\tilde{X}^*) = Ad(g^{-1})(\tilde{\theta}(\tilde{X}^*, X)).$$

Thus, $R_g^* \, \tilde{\omega} = Ad(g^{-1})\tilde{\omega}$ and $\tilde{\omega}(A^{\#}) = 0$, i.e., $\tilde{\omega}$ is a connection form in $Q(M, GL(n;\mathbb{R}), \tilde{\pi})$. From the third and the fourth properties above, (ii) in this lemma is evident. □

By the help of Lemma 2.1, we obtain the following proposition.

PROPOSITION 2.2. *The horizontal lift X^* of $X \in \mathfrak{X}(M)$ for the connection form $\tilde{\omega}$ in the principal fibre bundle $Q(M, GL(n;\mathbb{R}), \tilde{\pi})$ is given as follows :*

$$X^* = (\tilde{X}^*, X) \in \mathfrak{X}(Q), \tag{2.3}$$

where $\tilde{X}^*$ is the horizontal lift of $\phi_* X$ for the connection form ω in the bundle of linear frames over N.

Proof. $\tilde{\pi}_*((\tilde{X}^*, X)) = X$ and $\tilde{\omega}((\tilde{X}^*, X)) = (\tilde{\phi}^*\omega)((\tilde{X}^*, X)) = \omega(\tilde{X}^*)$ $= 0$. So, the proof of this proposition is complete. □

On the product manifold $Q \times \mathbb{R}^n$, we let $GL(n;\mathbb{R})$ act on the right as follows : an element $a \in GL(n;\mathbb{R})$ maps $((u,x),\xi) \in Q \times \mathbb{R}^n$ into

$$((u,x)a, a^{-1}\xi) = ((ua, x), a^{-1}\xi) \in Q \times \mathbb{R}^n.$$

The quotient space of $Q \times \mathbb{R}^n$ by this group action is denoted by

$$Q \times_{GL(n;\mathbb{R})} \mathbb{R}^n =: \phi^{-1}(TN) =: E.$$

For the horizontal lift $X^* = (\tilde{X}^*, X)$ of $X \in \mathfrak{X}(M)$ for the connection form $\tilde{\omega}$ in the principal fibre bundle $Q(M, GL(n;\mathbb{R}), \tilde{\pi})$, by virtue of Lemma 2.1 we easily obtain the following lemma.

LEMMA 2.3. *$\tilde{\theta}(X^*)$ corresponds to $\phi_*(X) \in \Gamma(M; \phi^{-1}(TN) = E)$, where*

$$(u,x) : \mathbb{R}^n \ni \xi \longmapsto u\xi \in \pi_E^{-1}(\{x\}) = T_{\phi(x)}N \qquad ((u,x) \in Q).$$

Proof. For $(u,x) \in Q$,

$$(\tilde{\theta}(X^*))((u,x)) = \theta_u(\tilde{X}^*) = u^{-1}(\phi_* X_*)$$

and

$$(u,x)(u^{-1}(\phi_* X_*)) = (\phi_* X)(\tilde{\pi}(u,x)) = (\phi_* X)(x).$$

□

And then, we get the following lemma.

LEMMA 2.4. *Let $\tau = x_t$ $(0 \le t \le 1)$ be a $C^\infty-$curve in M, and u_t $(\tilde{\tau}; (= \pi(u_t) = \phi(x_t) =: y_t$ $(0 \le t \le 1))$ the horizontal curve of the curve y_t for the connection form ω in $L(N)$. Then, $\gamma_t := (u_t, x_t) \in Q$ $(0 \le t \le 1)$ is the horizontal curve passing through $\gamma_0 \in Q$, of x_t for the connection form $\tilde{\omega}$ in $Q(M, GL(n;\mathbb{R}), \tilde{\pi})$.*

Proof. For each $t \in [0,1]$, $\tilde{\pi}(u_t, x_t) = x_t$ and $\tilde{\omega}(\dot{\gamma}_t) = \omega(\dot{u}_t) = 0$. □

For later use, we get the following lemma.

LEMMA 2.5. *Let $P(M, G, \pi)$ be a principal fibre bundle over a $C^\infty-$manifold M with structure group G over a $C^\infty-$manifold M, $\mathfrak{g}$ the Lie algebra of G, and $A^\#$ $(A \in \mathfrak{g})$ the fundamental vector field on P which corresponds to A. Then, $[A^\#, B^\#] = [A,B]^\#$ $(A, B \in \mathfrak{g})$.*

Proof. Let $a_t := \exp(tA)$ $(t \in \mathbb{R}, A \in \mathfrak{g})$ be a 1-parameter subgroup of G. Then, for each $u \in P$

$$[A^\#, B^\#]_u = \lim_{t \to 0} \frac{B^\#_u - ((R_{a_t})_* B^\#)_u}{t}.$$

Since $(R_{a_t})_* B^\# = (Ad(a_t^{-1})B)^\# = ((\exp(-t\ ad(A)))B)^\#$, we have

$$[A^\#, B^\#] = [A,B]^\#.$$

□

In general, the torsion form Θ for a connection form ω in the bundle $L(M)$ of all linear frames over M is defined by $\Theta = D\theta = d\theta \circ h$, where θ is the canonical form in $L(M)$.

We obtain the following structure equation for the connection form $\tilde{\omega} = \tilde{\phi}^* \omega$ in the principal fibre bundle $Q(M, GL(n;\mathbb{R}), \tilde{\pi})$ $(Q = \phi^{-1}(L(N)))$. Moreover, $d\tilde{\theta} \circ h$ in the induced bundle $\phi^{-1}(L(N)) = Q$ is said to be the *torsion form* in $Q(M, GL(n;\mathbb{R}), \tilde{\pi})$.

THEOREM 2.6 (structure equation). *Let ω be a connection form in $L(N)$, and θ the canonical form in $L(N)$. For $\tilde{\theta} := \tilde{\phi}^*\theta$ on $Q = \phi^{-1}(L(N))$, we obtain*

$$d\tilde{\theta} = -\tilde{\omega} \wedge \tilde{\theta} + d\tilde{\theta} \circ h. \tag{2.4}$$

Proof. Every vector of Q is a sum of a vertical vector and a horizontal vector. Since both sides of the above equality are bilinear and skew-symmetric in $\mathfrak{X}(Q)$, it is sufficient to verify the equality in the following three special cases.

(1) For $X^* = (\tilde{X}^*, X)$ and $Y^* = (\tilde{Y}^*, Y)$ $(X, Y \in \mathfrak{X}(M))$, we have $\tilde{\omega}(X^*) = 0$. So, the equality (2.4) is verified.

(2) For $X^*_{(u,x)} = (\tilde{X}^*_x, X_x)$ and $A^{\#}_{(u,x)}$ $(X_x \in T_xM, \tilde{X}_x = \phi_{*x}(X), A \in \mathfrak{gl}(n;\mathbb{R}), (u,x) \in Q)$,

$$\begin{aligned}
A^{\#}_{(u,x)}(\tilde{\theta}(X^*)) &= \lim_{t\to 0} \frac{(\tilde{\theta}(X^*))(u\exp(tA), x) - (\tilde{\theta}(X^*))(u,x)}{t} \\
&= \lim_{t\to 0} \frac{(\theta(\tilde{X}^*))(u\exp(tA)) - (\theta(\tilde{X}^*))(u)}{t} \\
&= \lim_{t\to 0} \frac{((\exp(tA))^{-1} \circ u^{-1})(\tilde{X}_x) - u^{-1}(\tilde{X}_x)}{t} \\
&= -A(u^{-1}(\tilde{X}_x))
\end{aligned}$$

and

$$\begin{aligned}
\left[A^{\#}, X^*\right]_{(u,x)} &= \left[(A^{\#}, O), (\tilde{X}^*, X)\right]_{(u,x)} \\
&= \lim_{t\to 0} \frac{(\tilde{X}^*, X)_{(u,x)} - (R_{\exp(tA)})_*(\tilde{X}^*, X)_{(u\exp(-tA),x)}}{t} \\
&= 0.
\end{aligned}$$

From these facts and Lemma 2.1, we get

$$\begin{aligned}
d\tilde{\theta}_{(u,x)}(X^*, A^{\#}) &= \frac{1}{2}\left\{X^*(\tilde{\theta}(A^{\#})) - A^{\#}(\tilde{\theta}(X^*)) - \tilde{\theta}\left(\left[X^*, A^{\#}\right]\right)\right\}_{(u,x)} \\
&= \frac{1}{2}A(u^{-1}(\tilde{X}_x)).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&(-\tilde{\omega} \wedge \tilde{\theta} \ + \ d\tilde{\theta} \circ h)_{(u,x)}(X^*, A^{\#}) \\
&= (-\tilde{\omega} \wedge \tilde{\theta})_{(u,x)}(X^*, A^{\#}) \\
&= \frac{1}{2}\left\{-\tilde{\omega}(X^*)\tilde{\theta}(A^{\#}) + \tilde{\omega}(A^{\#})\tilde{\theta}(X^*)\right\}_{(u,x)}
\end{aligned}$$

$$= \frac{1}{2} A(u^{-1}(\tilde{X}_x)).$$

So, the above equality (2.4) in this case is verified.

(3) For $A^{\#}_{(u,x)}$ and $B^{\#}_{(u,x)}$ $((u,x) \in Q; A, B \in \mathfrak{gl}(n;\mathbb{R}))$, by virtue of Lemma 2.5 we obtain

$$\left[A^{\#}, B^{\#}\right]_{(u,x)} = \left[(A^{\#},0),(B^{\#},0)\right]_{(u,x)} = \left([A,B]^{\#},0\right)_{(u,x)}.$$

So, the left side and the right side in the above equality (2.4) vanish. □

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